$$M_{+\pm} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = j \begin{bmatrix} \frac{\alpha_1 L}{\sqrt{2}} + \alpha_2 L \pm j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \pm\frac{2j\alpha_1 L}{\sqrt{2}} \pm 2j\frac{\alpha_2 L}{\sqrt{2}} + \sqrt{2} - \frac{1}{\sqrt{2}} & \frac{\alpha_1 L}{\sqrt{2}} + \alpha_2 L \pm j\frac{1}{\sqrt{2}} \end{bmatrix} \tag{13}$$

now,

$$\Gamma = \frac{A + B - C - D}{A + B + C + D} = \text{voltage reflection coefficient} \tag{14}$$

$$\Gamma_{+\pm} = \frac{\frac{2}{\sqrt{2}} - \sqrt{2} \mp \frac{2j\alpha_1 L}{\sqrt{2}} \mp 2j\alpha_2 L}{\sqrt{2} \pm j\frac{2}{\sqrt{2}} + \frac{2\alpha_1 L}{\sqrt{2}} + 2\alpha_2 L \pm \frac{2j\alpha_1 L}{\sqrt{2}} \pm 2j\alpha_2 L} \cdot \tag{15}$$

Letting A_4 equal the vector amplitude of the signal emerging from port 4,

$$A_4 = \tfrac{1}{2}[\Gamma_{++} - \Gamma_{+-}]. \tag{16}$$

Substituting (16) into (15) and discarding all second-order terms involving $\alpha_1 L$ and $\alpha_2 L$, it can be shown that

$$A_4 = -j\left[\frac{\alpha_1 L + \sqrt{2}\,\alpha_2 L}{2 + 4\alpha_1 L + 4\sqrt{2}\,\alpha_2 L}\right]. \tag{17}$$

Now

$$I = 20\log_{10}\left|\frac{1}{A_4}\right| \tag{18}$$

where I = isolation is dB

$$I = 20\log_{10}\left[\frac{2 + 4\alpha_1 L + 4\sqrt{2}\,\alpha_2 L}{\alpha_1 L + \sqrt{2}\,\alpha_2 L}\right]$$

$$\cong 20\log_{10}\left[\frac{2}{\alpha_1 L + \sqrt{2}\,\alpha_2 L}\right]. \tag{19}$$

III. "Rat Race" Hybrid Ring

The "rat race" hybrid ring (see Fig. 2) uses three quarter-wave (i.e., $\lambda/4$) transmission lines and one three-quarter-wave (i.e., $\frac{3}{4}\lambda$) transmission line. The same method of analysis previously used for the lossy square hybrid is directly applicable.

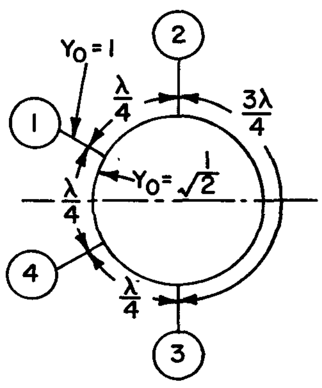

Fig. 2 "Rat race" hybrid ring.

Since the first shunt stub of the bisected network is an eighth-wavelength long, (7), (8), (9), (10), and (11) are applicable when $Y_0 = 1/\sqrt{2}$ and $\alpha_1 = \alpha$. Then

$$Y_{1+\pm} = \frac{\alpha L}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}; \tag{20}$$

for the second shunt stub

$$Y_{2++} = Y_0 \tanh\left(\frac{3\gamma L}{2}\right). \tag{21}$$

Letting $Y_0 = 1/\sqrt{2}$ and substituting (5) into (21),

$$Y_{2++} = 3\alpha L - j \tag{22}$$

now

$$Y_{2+-} = \frac{Y_0}{\tanh\left(\frac{3\gamma L}{2}\right)} \cdot \tag{23}$$

Letting $Y_0 = 1/\sqrt{2}$ and substituting (5) into (23)

$$Y_{2+-} = \frac{1}{\sqrt{2}}\left(\frac{1}{3\alpha L - j}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{3\alpha L + j}{(3\alpha L)^2 + 1}\right). \tag{24}$$

For small dissipation,

$$(3\alpha L)^2 \ll 1;$$

then

$$Y_{2+-} \cong \frac{3\alpha L}{\sqrt{2}} + \frac{j}{\sqrt{2}} \cdot \tag{25}$$

The *ABCD* matrix for the "rat race" hybrid ring can be evaluated using (20), (22), and (25) for $Y_{1+\pm}$ and $Y_{2+\pm}$. Upon determining the vector amplitude of the signal emerging from port 3 and discarding all second-order terms involving αL, it can be shown that

$$I = 20\log_{10}\left[\frac{4 + 12\sqrt{2}\,\alpha L}{\sqrt{2}\,\alpha L}\right]$$

$$\cong 20\log_{10}\left[\frac{2.83}{\alpha L}\right]. \tag{26}$$

IV. Numerical Results

The unloaded Q of a resonant length of lossy transmission line can be related conveniently to the attenuation per unit length when the dissipation is small:[4]

$$Q = \frac{\beta}{2\alpha} = \frac{\pi}{\lambda\alpha}, \tag{27}$$

where λ = wavelength. Rearranging terms and letting $L = \lambda/4$, it can be shown that

$$\alpha L = \frac{\pi}{4Q} = \frac{0.785}{Q} \cdot \tag{28}$$

[4] E. C. Jordan, *Electromagnetic Waves and Radiating Systems*. Englewood Cliffs, N. J.: Prentice-Hall, 1950, pp. 236–239.

Using (19), (26), and (28), theoretical hybrid isolations have been calculated for Q's of 10, 100, 1000, and ∞. These numerical results are tabulated below:

Q	L	I (Square Hybrid)	I ("Rat Race" Hybrid Ring)
10	0.0785	23.3 dB	33.7 dB
100	0.00785	40.8 dB	51.4 dB
1000	0.000785	60.5 dB	71.1 dB
∞	0	dB	dB

(Note: $\alpha_1 = \alpha_2 = \alpha$ is assumed for the square hybrid.)

For the square hybrid when $\alpha_1 = \alpha_2 = \alpha$

$$I = 20\log_{10}\left[\frac{2 + 9.66\alpha L}{2.414\alpha L}\right]$$

$$\cong 20\log_{10}\left[\frac{0.83}{\alpha L}\right]. \tag{29}$$

Upon comparing (26) and (29), it can be seen that as αL approaches zero, the isolation of the "rat race" hybrid ring will be 10.6 dB greater than the isolation of the square hybrid for the same αL in both hybrids. Such a performance advantage is not unexpected since the bandwidth of the square hybrid is less than that of the "rat race" hybrid ring.[1] Isolation in hybrid circuits is similar to peak rejection in a band-reject filter. As filter bandwidths become wider, the same amount of incidental dissipation (i.e., same resonator unloaded Q) results in higher peak rejection.

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A Stepped Mode Transducer Using Homogeneous Waveguides

***Abstract*—A rectangular to cylindrical waveguide transducer is described which couples the dominant rectangular (TE_{10}) and dominant cylindrical ($TE_{11}°$) modes. The maximum voltage reflection coefficient remains less than 0.025 over the design bandwidth. Symmetry considerations substantiated by moding tests show the transducer to be higher-order mode free. Previous work is reviewed, the design method discussed, and experimental data shown.**

Broadband rectangular to dominant mode cylindrical waveguide transducers are common to several devices in the microwave region, most notable of which is perhaps the precision rotary vane attenuator. Frequently such transducers are realized by construction of a taper section several wavelengths in size. As a result of a recent study of the dominant

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mode cutoff wavelengths of truncated cross-section cylindrical waveguide, Pyle [1] has shown the conditions necessary to realize a transition of variable cross section but constant cutoff wavelength between the waveguides. Pyle's conditions may be applied to the tapered line approach; the resulting structure would be classed homogeneous since the guide wavelength would be independent of position along the direction of propagation.

Stuchly and Kraszewski [2] have recently reported a stepped transducer based on a series of E-plane truncated cylindrical waveguide quarter-wave sections with a single H-plane step to rectangular cross section. The design basis was equality of cylindrical and rectangular waveguide cutoff wavelengths yielding an overall frequency independent impedance transformation ratio. Because the cutoff wavelength of E-plane truncated cross-section cylindrical waveguide depends upon the degree of truncation [3], quarter-wave sections of this type introduce frequency dependent impedance ratios thereby causing this structure to be classed inhomogeneous.

A completely homogeneous stepped solution is possible by retaining the cutoff wavelength equality between the cylindrical and rectangular waveguides as well as among the quarter-wave transformer sections by applying Pyle's conditions for constant cutoff wavelength at selected impedance levels. Pyle's conditions lead to simultaneous E- and H-plane truncations of the cylindrical waveguide; this approach not only avoids possible problems of frequency dependent impedance ratios but also distributes the H-plane step among the junctions.

Avoiding the question of a consistent solution for the characteristic impedance of E- and H-plane truncated cylindrical waveguide, it was assumed, since λc is constant for all sections, that characteristic impedance

$$Z_0 \propto 2\frac{b}{a} \tag{1}$$

where a, b are defined in Fig. 1. Note that, in the limits, $b/a=1/2$, $b=a=2r$, (1) is rigorously consistent with the power-voltage definitions of characteristic impedance for dominant moded rectangular and cylindrical waveguides [4]. The approximation made here is that (1) applies for the intermediate E- and H-plane truncated sections, $1/2<b/a<1$.

Young's tables for homogeneous quarter-wave transformer [5] were used to obtain characteristic impedance levels for a selected bandwidth $w=0.80$ and an impedance transformation ratio of 2.000 with $N=4$ sections. Table I lists the impedance levels and b/a ratios which simultaneously satisfy (1) and the cutoff wavelength conditions [1]. Junction susceptance length corrections were obtained for the E- and H-planes independently similar to Young's outline [6] and are shown in Table I.

The completed transducer is shown in Fig. 2. In order to test the design assumptions under controlled conditions and to insure a high degree of symmetry, tolerances better than $10^{-3}\lambda_0$ were held.

Transducer VSWR was obtained in WR-430 slotted section and sliding cylindrical termination with spot checks using a high accuracy tuned reflectometer. Three experiments were run to assess optimum junction susceptance length corrections. The first run used length corrections consisting of the algebraic sum of the E- and H-plane length corrections listed in Table I. A final run consisted of E-plane length corrections only. An intermediate run, based on weighting the length corrections according to E- and H-plane susceptance magnitudes gave best performance. The weighted lengths are listed in Table I and the results shown in Fig. 3. Predicted performance based on absence of frequency dependent junction susceptances is a Tchebyscheff ripple voltage reflection coefficient maximum of 0.008 over 1680–2620 MHz. The results show a mean reflection coefficient approximately twice theoretical with slight bandwidth compression.

No evidence of possible $TM_{01}{}^{\circ}$ nor $TE_{21}{}^{\circ}$ moding was detected with the transducer exciting a mode sensitive suppressed sidelobe conical feedhorn. The moding detector was the space radiation patterns of the feedhorn.

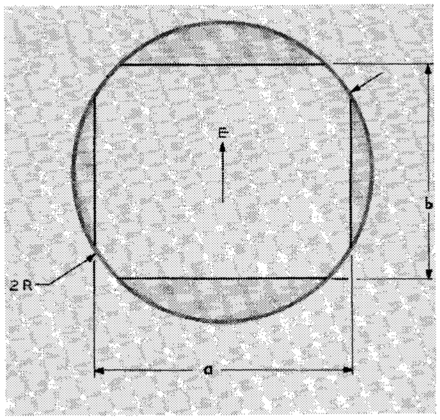

Fig. 1. E- and H-plane truncated cylindrical waveguide.

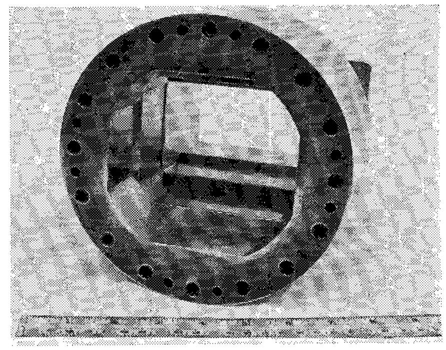

Fig. 2. Homogeneous waveguide transducer.

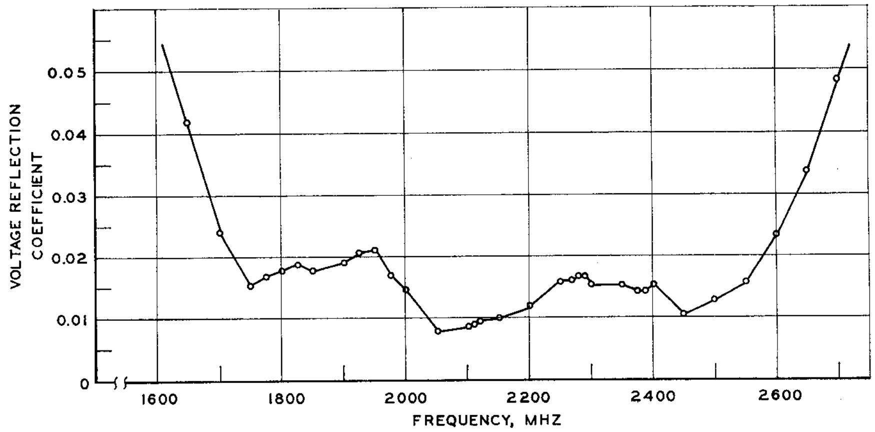

Fig. 3. Reflection coefficient, WR-430—5.040 inch diameter transducer.

TABLE I

Impedance Levels, Cross Sections and Lengths 1700—2600 MHz Transducer

Section	Normalized Characteristic Impedance	Type	b Inches	a Inches	Capacitive Junction Correction Inches	Inductive Junction Correction Inches	Weighted Length, Inches $\lambda g_0/4$ =1.844 inch
Z_0	1.00000	Rectangular	2.150	4.300	—	—	—
Z_1	1.06726	Rectangular	2.295	4.300	−0.073	0.000	1.771
Z_2	1.26420	Rectangular	2.720	4.303	−0.112	+0.027	1.732
Z_3	1.58203	Truncated, $2r=5.040''$	3.468	4.384	−0.091	+0.078	1.781
Z_4	1.87396	Truncated, $2r=5.040''$	4.346	4.638	−0.003	+0.029	1.841
Z_5	2.00000	Cylindrical, $2r=5.040''$	5.040	5.040	—	—	—

Acknowledgment

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lar to circular waveguide mode and impedance transformer," *IEEE Trans. on Microwave Theory and Techniques* (*Correspondence*), vol. MTT-13, pp. 379–380, May 1965.

[3] J. R. Pyle and R. J. Angley, "Cutoff wavelengths of waveguides with unusual cross sections," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 556–557, September 1964.

[4] G. C. Southworth, *Principles and Applications of Waveguide Transmission*. Princeton, N. J.: Van Nostrand, 1964, pp. 104–125. (Note p. 104 typographical transpose of 754 to 574 ohms.)

[5] L. Young, "Tables for cascaded homogeneous quarter-wave transformers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-7, pp. 233–237, April 1959. (Correction: L. Young, *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-8, pp. 243–244, March 1960.)

[6] L. Young, "Practical design of a wide-band quarter-wave transformer in waveguide," *Microwave J.*, vol. 6, pp. 76–79, October 1963.

An Approximate Formula for Calculating Z_0 of a Symmetric Strip Line

The equations available to determine the characteristic impedance of a symmetric strip line (Fig. 1) to high accuracy are difficult to utilize without a computer [1], [2]. Other less accurate equations which are easier to apply have been developed, and the one most frequently quoted has been given by Cohn [3]:

$$Z_0 = \frac{94.15/\sqrt{\epsilon_r}}{[W/(D-T) + C_0/\pi]} \tag{1}$$

where

$$C_0 = \frac{2}{1-T/D} \ln\left[\frac{1}{1-T/D} + 1\right] - \left[\frac{1}{1-T/D} - 1\right] \cdot \ln\left[\frac{1}{(1-T/D)^2} - 1\right]. \tag{2}$$

Equation (1) was stated by Cohn to be applicable over the range $W/(D-T) \geq 0.35$ and $T/D \leq 0.25$, with a maximum error of approximately 1 percent at the lower limit of W/D. Good agreement between computed and measured values of Z_0 has also been obtained for values of W/D and T/D outside Cohn's stated limits [4].

Chen [5] has supplied another equation to determine Z_0 for a symmetric strip line:

$$Z_0 = \frac{94.15/\sqrt{\epsilon_r}}{[W/(D-T) + C_e/\pi} \tag{3}$$

where

$$C_e = \frac{D}{D-T} \ln\left[\frac{2D-T}{T}\right] + \ln\left[\frac{T(2D-T)}{(D-T)^2}\right]. \tag{4}$$

Although not immediately obvious, and apparently not realized by some [4], (2) and (4) are identical. By substituting $x = 1/(1-T/D)$,

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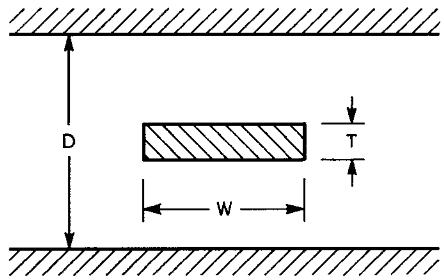

Fig. 1. Cross section of symmetric slab line.

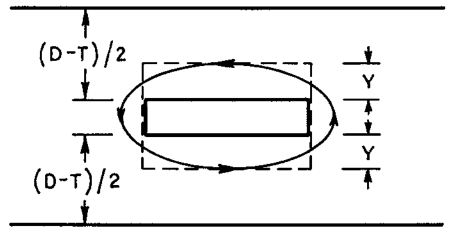

Fig. 2. Approximation to line of magnetic field intensity.

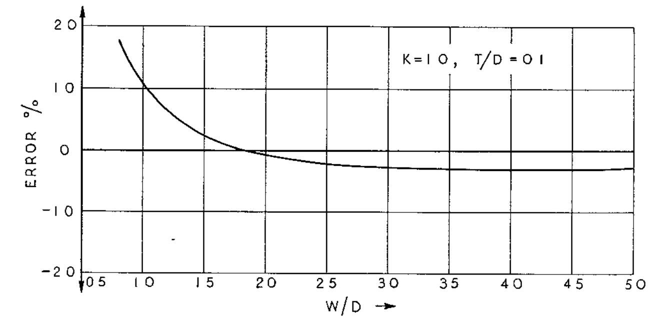

Fig. 3. Percentage error in characteristic impedance.

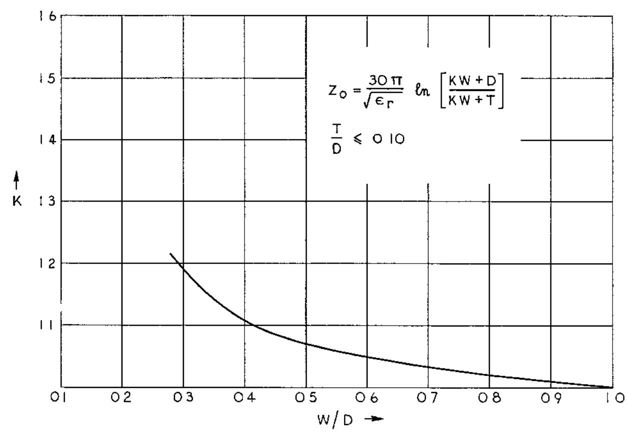

Fig. 4. Variation of correction factor with W/D ratio.

$$C_0 = 2x \ln(x+1) - (x-1)\ln(x^2-1) = \ln\left[\frac{(x+1)^{x+1}}{(x-1)^{x-1}}\right].$$

From (4),

$$C_e = \frac{1}{1-T/D} \ln\left[\frac{2-T/D}{T/D}\right] + \ln\left[\frac{(T/D)(2-T/D)}{(1-T/D)^2}\right].$$

Now $T/D = 1 - 1/x$ and $2 - T/D = 1 + 1/x$.

$$\therefore\ C_e = x \ln\left[\frac{1+1/x}{1-1/x}\right] + \ln\left[(1-1/x)(1+1/x)x^2\right] = \ln\left[\frac{(x+1)^{x+1}}{(x-1)^{x-1}}\right].$$

Hence, $C_0 = C_e$ and (1)–(4) can be combined together in the following expression:

$$Z_0 = \frac{94.15}{\sqrt{\epsilon_r}\left[\frac{W}{D}\cdot x + \frac{1}{\pi}\ln\left\{\frac{(x+1)^{x+1}}{(x-1)^{x-1}}\right\}\right]}. \tag{5}$$

However, (5) is not a particularly easy one to handle, and a simpler although less comprehensive one has been developed, accurate to better than 1.2 percent of (5) for $W/D \geq 1.0$, and to within 5 percent for $W/D \geq 0.75$, providing that in both cases $T/D \leq 0.2$. It is

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_r}} \ln\left(\frac{W+D}{W+T}\right). \tag{6}$$

Equation (6) can be obtained by considering the "average" length of the lines of magnetic field intensity surrounding the central conductor of a strip line. The characteristic impedance of a uniform transmission line operating in the transverse electromagnetic (TEM) mode can be determined by first calculating the inductance per unit length of the line L then applying the equation

$$Z_0 = c_0'L \tag{7}$$

where c_0' is the velocity of a TEM wave in an infinite medium of dielectric, relative permittivity ϵ_r.